University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 6

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1. K&K problem 6.1

(a.) We know that the total linear momentum of the system is zero. (This would occur, for example, if we were in the center of mass frame.)

$$\mathbf{P} = \sum_i \mathbf{p}_i = \mathbf{0}$$

Examine the total angular momentum of the system. The vector from the origin to point i is denoted $\mathbf{r}_i$. The angular momentum in general depends on where the origin is:

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$$

We now want to find the angular momentum about a new origin whose position vector is $\mathbf{R}$ in the current coordinate system. In this new system, the position vector of point i becomes $\mathbf{r}_i - \mathbf{R}$. Each point has its position changed by the same amount. The new value of the angular momentum is

$$\mathbf{L}_{\text{new}} = \sum_i (\mathbf{r}_i - \mathbf{R}) \times \mathbf{p}_i$$

Expanding,

$$\mathbf{L}_{\text{new}} = \sum_i \mathbf{r}_i \times \mathbf{p}_i - \sum_i \mathbf{R} \times \mathbf{p}_i$$

Because $\mathbf{R}$ is the same for all points, we can pull it outside of the sum:

$$\begin{aligned}\mathbf{L}_{\text{new}} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i - \mathbf{R} \times \sum_i \mathbf{p}_i \\ &= \sum_i \mathbf{r}_i \times \mathbf{p}_i - \mathbf{R} \times \mathbf{P}\end{aligned}$$

We know that $\mathbf{P} = \mathbf{0}$, so we are done.

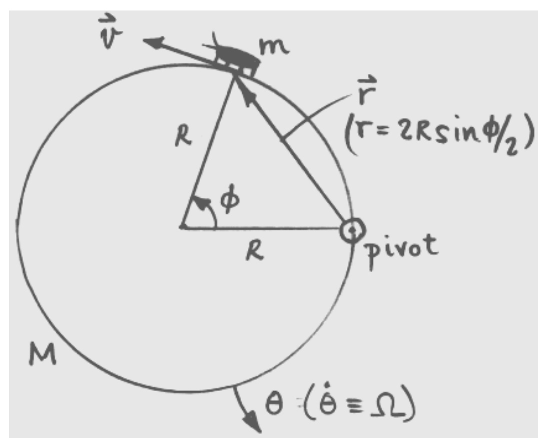
$$\mathbf{L}_{\text{new}} = \mathbf{L}$$

(b.) The proof for this part is identical if angular momentum is replaced by torque and linear momentum is replaced by force.

2. K&K problem 6.3

This problem and the next concern the same system – that of a bug walking along a hoop that is free to pivot around a point on its edge. The hoop lies flat on a frictionless surface. The ring has mass M and radius R, and the bug has mass m and walks on the ring with speed v.

The key idea in this problem is conservation of angular momentum. About the pivot there is no net torque on the system, so the total angular momentum about that point is conserved. The ring starts at rest with the bug on the pivot, and the bug starts walking at speed v. Immediately after the bug starts walking, the total angular momentum measured about the pivot point continues to be zero. The ring is not yet moving, so it has no angular momentum; the bug has begun to move, but it is at $\mathbf{r} = 0$, so it has no angular momentum yet.



We want to find the angular velocity Ω of the ring when the bug is opposite to the pivot. The bug is moving at speed v on the ring, but the ring is also moving. The bug is at a distance $2R$ from the pivot, so the velocity of that portion of the ring which is under the feet of the bug is $2\Omega R$. The total velocity of the bug is thus $v + 2\Omega R$. Next we need to know the moment of inertia of the hoop. A hoop has moment of inertia $I = MR^2$ about its center of mass. We use the parallel axis theorem to find the moment of inertia about a point on the edge.

$$I = I_{\mathrm{CM}} + Md^2$$

The distance d from the center of mass to the desired axis in this case is just R, so the moment of inertia of the hoop about a point on the edge is $I = 2MR^2$. We can now find an expression for the total angular momentum of the system. For the hoop we use $L = I\Omega$ and for the bug we use $L = mvr\sin\theta$. The angle θ between the position vector and the velocity vector of the bug in this case is simply $\pi/2$, so $\sin\theta$ is just 1. We now write the angular momenta of the two pieces

$$L_{\mathrm{bug}} = 2mR(v + 2\Omega R) \qquad L_{\mathrm{hoop}} = 2MR^2\Omega$$

Since angular momentum about the pivot is conserved throughout the motion, We know that $L_{\mathrm{bug}} + L_{\mathrm{hoop}} = 0$. This gives the following expression:

$$2mvR + 4m\Omega R^2 + 2MR^2\Omega = 0$$

We solve this equation for Ω in terms of v and get

$$\Omega = -\frac{mv}{MR + 2mR}$$

Note that the minus sign means that the hoop rotates in a direction opposite to that in which the bug moves. This makes sense because the total angular momentum about the pivot point must vanish.

3. We now study the bug and hoop system in more detail. See the diagram in the previous problem.

(a.) In this part we assume that the ring is fixed. We want to calculate the angular momentum of the bug about the pivot point. The first step is to find the distance r between the bug and the pivot. We can do this using the law of cosines. Consider the triangle made by the line between the bug and the pivot, and the two radial lines extending from the center of the hoop to the bug and pivot, respectively. This is an isosceles triangle, with two equal sides of length R having an angle ϕ between them. If we define the azimuth of the bug on the hoop to be zero at the pivot, the angle ϕ is simply the azimuth of the bug on the hoop. The length r of the third side is found using the law of cosines:

$$r^2 = R^2 + R^2 - 2R^2\cos\phi \;\Rightarrow\; r^2 = 2R^2(1 - \cos\phi)$$

Using the trigonometric identity $1 - \cos\phi = 2\sin^2\frac{\phi}{2}$, we can get a simple result for r:

$$r = 2R\sin\frac{\phi}{2}$$

Now we know v and r. The only thing left to determine is the angle between the position and velocity vectors. The first step is to find the angle between the position vector $\mathbf{r}$ of the bug and the line from the center of the circle to the bug. The isosceles triangle (like any triangle) has a total angle π, and its central angle is ϕ. The remaining two angles are equal, so they must be $(\pi - \phi)/2$ each. Thus the three angles add up to π radians. Because the velocity vector $\mathbf{v}$ is tangent to the circle, the angle between $\mathbf{r}$ and $\mathbf{v}$ is $\pi/2$ minus this angle. Thus the angle between $\mathbf{r}$ and $\mathbf{v}$ is $\phi/2$. We can now get the angular momentum of the bug. Assuming that the ring is fixed, $l = mvr\sin\theta$ from the bug alone, so

$$l = 2mvR\sin^2\frac{\phi}{2}$$

(b.) In this part we assume that the ring is rotating with angular velocity Ω, but that the bug is fixed on the ring. The velocity of the bug is just Ωr, where r is the same as was calculated in the previous part. The velocity in this case is always perpendicular to its position vector. This can be seen by remembering that the bug isn't moving on the ring, so it must be in uniform circular motion about the pivot, with a velocity that is tangent to its present position. Therefore the angular momentum l' of the bug is simply mvr, yielding

$$l' = m\Omega r^2 = 4m\Omega R^2\sin^2\frac{\phi}{2}$$

(c.) We now allow both the bug and the ring to move. The total angular momentum of the bug is $l+l'$ from parts (a.) and (b.) respectively. To this we must add the angular momentum of the ring to get the total angular momentum of the system. From problem **2.** we know that the total angular momentum must be zero. The angular momentum of the ring is $I\Omega$, so we get the following equation.

$$4m\Omega R^2 \sin^2\frac{\phi}{2} + 2mvR\sin^2\frac{\phi}{2} + 2MR^2\Omega = 0$$

We solve this for Ω in terms of ϕ. The result is

$$\Omega = -\frac{mv\sin^2\frac{\phi}{2}}{MR + 2mR\sin^2\frac{\phi}{2}}$$

This agrees with the result of problem **2.** when the bug is at $\phi = \pi$, opposite to the pivot.

(d.) Finally, we want to find an expression for the angle θ through which the ring rotates. We know that θ is related by a simple differential equation to the angular velocity Ω of the hoop

$$\Omega = \frac{d\theta}{dt}$$

but we want to express Ω in terms of ϕ so that we can use the fact that $Rd\phi/dt$, the speed v of the bug with respect to the rim of the hoop, is constant. We apply the chain rule to get

$$\Omega = \frac{d\theta}{dt} = \frac{d\theta}{d\phi}\frac{d\phi}{dt}$$

Substituting $d\phi/dt = v/R$, where v is constant, we can write an integral for θ:

$$\theta = -\frac{R}{v}\int_{\phi_o}^{\phi} \frac{mv\sin^2(\phi'/2)}{MR + 2mR\sin^2(\phi'/2)} d\phi'$$

We can simplify this a little, but doing the integral is hard, which is why you weren't asked to evaluate it. Setting the initial bug azimuth ϕ_0 to zero and using the fact that $d\phi/dt = v/R$ is a constant so that $\phi = vt/R$,

$$\theta(t) = -\int_0^{vt/R} \frac{\sin^2(\phi'/2)}{(M/m) + 2\sin^2(\phi'/2)} d\phi'$$

4. K&K problem 6.5

A car of mass m is parked on a slope of angle θ facing uphill. The center of mass is a distance d above the ground, and it is centered between the wheels, which are a distance l apart. We want to find the normal force exerted by the road on the front and rear tires.

It is easiest to do this problem choosing the origin as the point on the road directly below the center of mass. About this point there are three torques. The normal force on the front (N_f) and rear (N_r) set of wheels provides a torque, and also gravity provides a torque $mgd\sin\theta$ because the car isn't horizontal. However, the forces of friction on the tires don't provide any torque because they are in line with the direction to the origin. The torque from the front wheels and the torque due to gravity tend to want to flip the car over backwards, while the torque on the rear wheels opposes this tendency. We want the sum of the torques to vanish, because the (static) car is not undergoing any acceleration, angular or linear:

$$\begin{aligned} 0 &= N_f\frac{l}{2} + mgd\sin\theta - N_r\frac{l}{2} \\ N_r - N_f &= \frac{2mgd}{l}\sin\theta \end{aligned}$$

We can get one more condition from the fact that the car is not undergoing linear acceleration perpendicular to the road. This means that the normal forces exactly cancel gravity:

$$N_r + N_f = mg\cos\theta$$

We can take the sum and difference of these two equations to get expressions for N_f and N_r. These are

$$\begin{aligned} N_r &= mg\Big(\frac{1}{2}\cos\theta + \frac{d}{l}\sin\theta\Big) \\ N_f &= mg\Big(\frac{1}{2}\cos\theta - \frac{d}{l}\sin\theta\Big) \end{aligned}$$

Plugging in $\theta = 30°$, $mg = 3000$ lb, $d = 2$ ft, and $l = 8$ ft, we get $N_r =$1674 lb and $N_f =$924 lb.

5. We will solve this problem symbolically and wait until the end to plug in numbers. This is always good practice because it makes it a lot easier to check the units of the result and to explore whether the result is reasonable when the inputs have limiting values. We take M to be the mass of the man ($Mg = 180$ lb) and m to be the mass of the ladder ($mg = 20$ lb). The length H of the ladder is 12 ft, and its point of contact with the wall is $d = 6$ ft from the wall. The angle that it makes with the wall is $\theta = \arcsin(d/H) = 30°$. Finally, the force of friction on the ladder from the ground is $F_f \leq F_f^{\max}$, where $F_f^{\max} = 80$ lb.

There are five forces to consider in this problem. They are the two normal forces on the ladder, N_g from the ground and N_w from the wall; the force F_f of friction at the base of the ladder; and the two forces of gravity, Mg on the man and mg on the ladder. This is a torque balance problem, so choosing a good origin makes it a lot easier. With this choice of the point of contact with ground, two of the five forces contribute no torque about that point. Not bad! As a sanity check we evaluate μ, the coefficient of friction between the ladder and the ground. The normal force N_f from the floor is equal and opposite to $(M+m)g$, the sum of the weights of the ladder and the man. We are given the maximum frictional force $F_f^{\max}$, and we know that $F_f^{\max} = \mu N$, so $\mu = F_f^{\max}/((M+m)g) = 80/200 = 0.4$, a reasonable value.

We now calculate the torques. To find the maximum height h to which the man can climb without the ladder slipping, we assume that the ladder is about to slip. This means that the normal force N_w from the wall is equal and opposite to $F_f^{\max}$, exactly countering the maximum force of friction: since these two forces are the only forces in the horizontal direction they must sum to zero. The torque from the wall is then $\tau_w = -F_f^{\max} H\cos\theta$, where the minus sign indicates that this torque pushes clockwise. The torque from the weight of the ladder is exerted at the midpoint of the ladder, its center of mass. The value of this torque is $\tau_m = mg\frac{H}{2}\sin\theta$. Similarly, the torque exerted by the weight of the man, who is a distance h up the ladder, is $\tau_M = Mgh\sin\theta$. Requiring these three torques to sum to zero,

$$\begin{aligned} 0 &= \tau_M + \tau_m + \tau_w \\ &= Mgh\sin\theta + mg\frac{H}{2}\sin\theta - F_f^{\max} H\cos\theta \end{aligned}$$

Solving for h,

$$\begin{aligned} h &= \frac{F_f^{\max} H\cos\theta - mg\frac{H}{2}\sin\theta}{Mg\sin\theta} \\ \frac{h}{H} &= \frac{F_f^{\max}}{Mg}\cot\theta - \frac{m}{2M} \\ &= \frac{\mu(M+m)}{M}\cot\theta - \frac{m}{2M} \\ &= \frac{\mu(M+m)\cot\theta - (m/2)}{M} \\ &= \frac{0.4(200)\sqrt{3} - 10}{180} \\ &= 0.7142 \\ h &= 8.571 \text{ ft} \end{aligned}$$

6. K&K problem 6.8

Because of the spherical symmetry, we work in spherical polar coordinates. To find the moment of inertia we need to evaluate the integral

$$I = \int r_\perp^2 \rho dv$$

where in these coordinates $r_\perp = r\sin\theta$ is the perpendicular distance to the axis and $dv = r^2 dr\, d(\cos\theta)\, d\phi$ is the element of volume. The integral to evaluate is thus

$$I = M\frac{\int_0^R r^4\, dr \int_{-1}^1 \sin^2\theta\, d(\cos\theta) \int_0^{2\pi} d\phi}{\int_0^R r^2\, dr \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi}$$

where the denominator is the volume V of the sphere, needed to evaluate its density $\rho = M/V$. Substituting $u \equiv r/R$,

$$\frac{I}{MR^2} = \frac{\int_0^1 u^4\, du \int_{-1}^1 \sin^2\theta\, d(\cos\theta) \int_0^{2\pi} d\phi}{\int_0^1 u^2\, du \int_{-1}^1 d(\cos\theta) \int_0^{2\pi} d\phi}$$

In both the numerator and the denominator, all three integrals have limits that do not depend on the other variables, so each integral can be evaluated independently. The ϕ integrals cancel, and the u integrals have the ratio

$$\frac{1/5}{1/3} = \frac{3}{5}$$

The integrand in the $\cos\theta$ integral in the numerator can be rewritten

$$\begin{aligned}\sin^2\theta\, d(\cos\theta) &= (1-\cos^2\theta)d(\cos\theta)\\ &= d(\cos\theta) - d\Big(\frac{1}{3}\cos^3\theta\Big)\end{aligned}$$

Therefore the ratio of the θ integrals is

$$\frac{2-\frac{2}{3}}{2} = \frac{2}{3}$$

Putting it all together,

$$I = MR^2 \times \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}MR^2$$

7. K&K problem 6.14

When the stick is released, there are two forces acting on it, gravity at the midpoint, and the normal force at the point B. We use the point B as the origin, so the only torque about this point is provided by gravity. At the moment of release, the stick is still horizontal, so the torque is

$$\tau_B = -\frac{Mgl}{2}$$

where the minus sign indicates that the torque pulls clockwise. We know that the moment of inertia of a thin stick about its endpoint is $I = Ml^2/3$, so we can easily find the angular acceleration α from $\tau = I\alpha$.

$$-\frac{Mgl}{2} = \frac{1}{3}Ml^2\alpha \;\Rightarrow\; \alpha = -\frac{3g}{2l}$$

The vertical acceleration of the center of mass is given by the simple formula $a = \alpha r$, where r is the distance between the center of mass and point b, about which the stick is (instantaneously) executing circular motion. (This is analogous to the expression $v = \omega r$.) Here this distance is $r = l/2$. This gives the acceleration of the center of mass:

$$a = -\frac{3}{4}g$$

where the minus sign indicates that the acceleration is downward. Finally we use Newton's second law to find the normal force at B. We know the acceleration and we know the force of gravity, so this is a simple equation

$$N - Mg = -\frac{3}{4}Mg \;\Rightarrow\; N = \frac{1}{4}Mg$$

where the positive direction is up, opposite to the force of gravity.

8. K&K problem 6.18

We want to find the equation of motion of the pendulum to determine the frequency. We will use the torque equation $\tau = I\alpha$. If we choose the pivot point of the pendulum as the origin, only one force provides torque, the force of gravity. It acts on the center of mass of the pendulum, a distance l_{cm} from the pivot point. The magnitude of this force is just $(M+m)g$. Thus the total torque is

$$\tau = -(M+m)gl_{cm}\sin\theta$$

where θ is the angular position of the pendulum. Writing the torque equation and approximating $\sin\theta \approx \theta$, we get

$$I\ddot{\theta} = -(M+m)gl_{cm}\theta$$

We recognize that this is the equation for a simple harmonic oscillator. The angular frequency and period are thus

$$\omega = \sqrt{\frac{(M+m)gl_{cm}}{I}} \qquad T = 2\pi\sqrt{\frac{I}{(M+m)gl_{cm}}}$$

All that is left is to evaluate l_{cm} and I. The equation for the center of mass is easy to use. The center of mass of the rod is halfway along its length, and the disk is a distance l from the pivot, so

$$l_{cm} = \frac{ml/2 + Ml}{M+m}$$

This expression simplifies the formula for the period

$$T = 2\pi\sqrt{\frac{I}{(M+m/2)gl}}$$

In the first case, where the disk is tied to the rod, the moment of inertia is determined using the parallel axis theorem. The disk is fixed to the rod, so, as the rod pivots, the disk must rotate at the same angular velocity. The moment of inertia of a stick about its end is $ml^2/3$. The moment of inertia of a disk about its center is $MR^2/2$. Because the center of mass is displaced a distance l from the origin, the parallel axis theorem tells us that the total moment of inertia of the disk is $MR^2/2 + Ml^2$. Thus the total moment of inertia of the pendulum is

$$I = \frac{1}{2}MR^2 + \left(M + \frac{1}{3}m\right)l^2$$

This gives the period of oscillation

$$T = T = 2\pi\sqrt{\frac{MR^2/2 + (M+m/3)l^2}{(M+m/2)gl}}$$

In the second case, where the disk is free to rotate on the rod, the moment of inertia is smaller. Because the disk is not fixed, it has no tendency to rotate. Its effective moment of inertia about the center of mass is thus zero. For our purpose it is the same as a point mass a distance l away from the origin. The new moment of inertia is

$$I = \left(M + \frac{1}{3}m\right)l^2$$

The period of oscillation in this case is also smaller:

$$T = 2\pi\sqrt{\frac{(M+m/3)l}{(M+m/2)g}}$$

This is the same as our answer for the first case in the limit $R \to 0$.